Super-Polynomial Growth of the Generalized Persistence Diagram

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Condensed Abstract

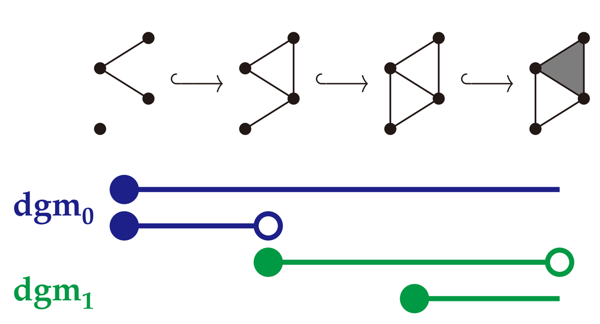
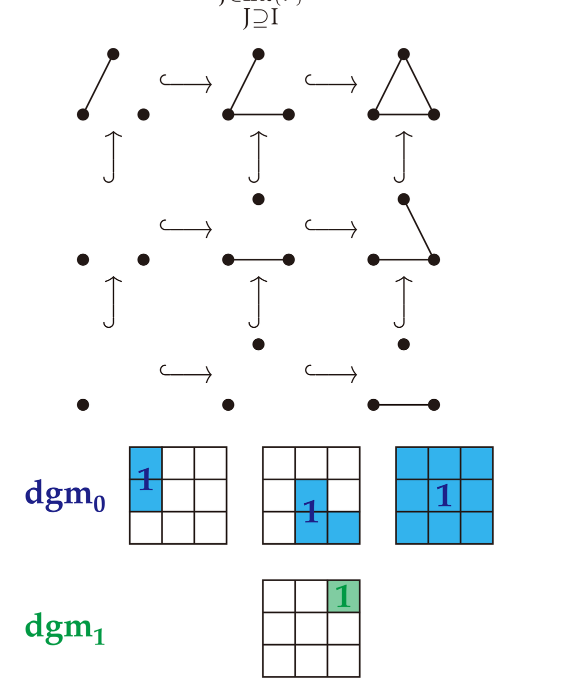
The Generalized Persistence Diagram (GPD) is a natural extension of the classical notion of persistence diagram, for multi-parameter persistence homology. However, little is known about the **size of the GPD** (the cardinality of the support of the GPD) despite its significance in the computation of the GPD, as it is defined to be the Möbius inverse of the Generalized Rank Invariant (GRI).

Our primary concern is to answer the question: *Given a d-parameter simplicial filtration, could the size of its GPD also be bounded by a polynomial in the number of simplices in the filtration?* The question arises from an observation in the one-parameter case ($d = 1$), where the number of simplices in a given filtration bounds the size of its GPD. If this were also the case for $d \geqslant 2$, it might be possible to compute the GPD directly and much more efficiently.

We show that the answer to the question above is negative, demonstrating the inherent difficulty of computing the GPD. More specifically, we construct a sequence of d-parameter simplicial filtrations where the cardinalities of their GPDs are not bounded by any polynomial in the number of simplices. Furthermore, we show that several commonly used methods for constructing multi-parameter filtrations can give rise to such "wild" filtrations.

The Generalized Persistence Diagram

The **Generalized Persistence Diagram (GPD)** [8, 11] is a generalization of the persistence diagram [6, 12] to the setting of multi-parameter persistent homology. Multi-parameter persistent homology is a generalization of (1-parameter) persistent homology, which allows for extracting finer and more detailed topological information of datasets. In the setting of (1-parameter) persistent homology, a dataset induces a filtration and a persistence module over $\mathbb{Z}$ or $\mathbb{R}$. In contrast, within the framework of multi-parameter persistent homology, a dataset induces a filtration and a persistence module over a partially ordered set (**poset**). Then, the GPD is defined as the Möbius inverse of the **Generalized Rank Invariant (GRI)** [4, 8, 9] on the set of intervals, analogous to the relation between the rank invariant [2] and the persistence diagram. Roughly speaking, the GRI counts the number of features that persist throughout an interval, and the GPD aims to count how many features persist exactly on a given interval, not beyond.

Persistent Homology	Multi-Parameter Persistent Homology
Rank Invariant $\mathrm{rk}_M([a,b]) = \mathrm{rank}(M_a \to M_b)$	**GRI** $\mathrm{rk}_M : \mathrm{Int}(P) \to \mathbb{Z}_{\geqslant 0}$ $\mathrm{rk}_M(I) := \mathrm{rank}(\varprojlim M\vert_I \to \varinjlim M\vert_I)$
Persistence Diagram $\mathrm{dgm}_M([a,b]) :=$ mult. of $\mathbb{I}_{[a,b]}$ in M $\mathrm{rk}_M([a,b]) = \sum_{\substack{J \in \mathrm{Int}(\mathbb{Z}) \\ J \supseteq [a,b]}} \mathrm{dgm}_M(J)$  	**GPD** $\mathrm{dgm}_M : \mathrm{Int}(P) \to \mathbb{Z}$ $\mathrm{rk}_M(I) = \sum_{\substack{J \in \mathrm{Int}(P) \\ J \supseteq I}} \mathrm{dgm}_M(J)$ 

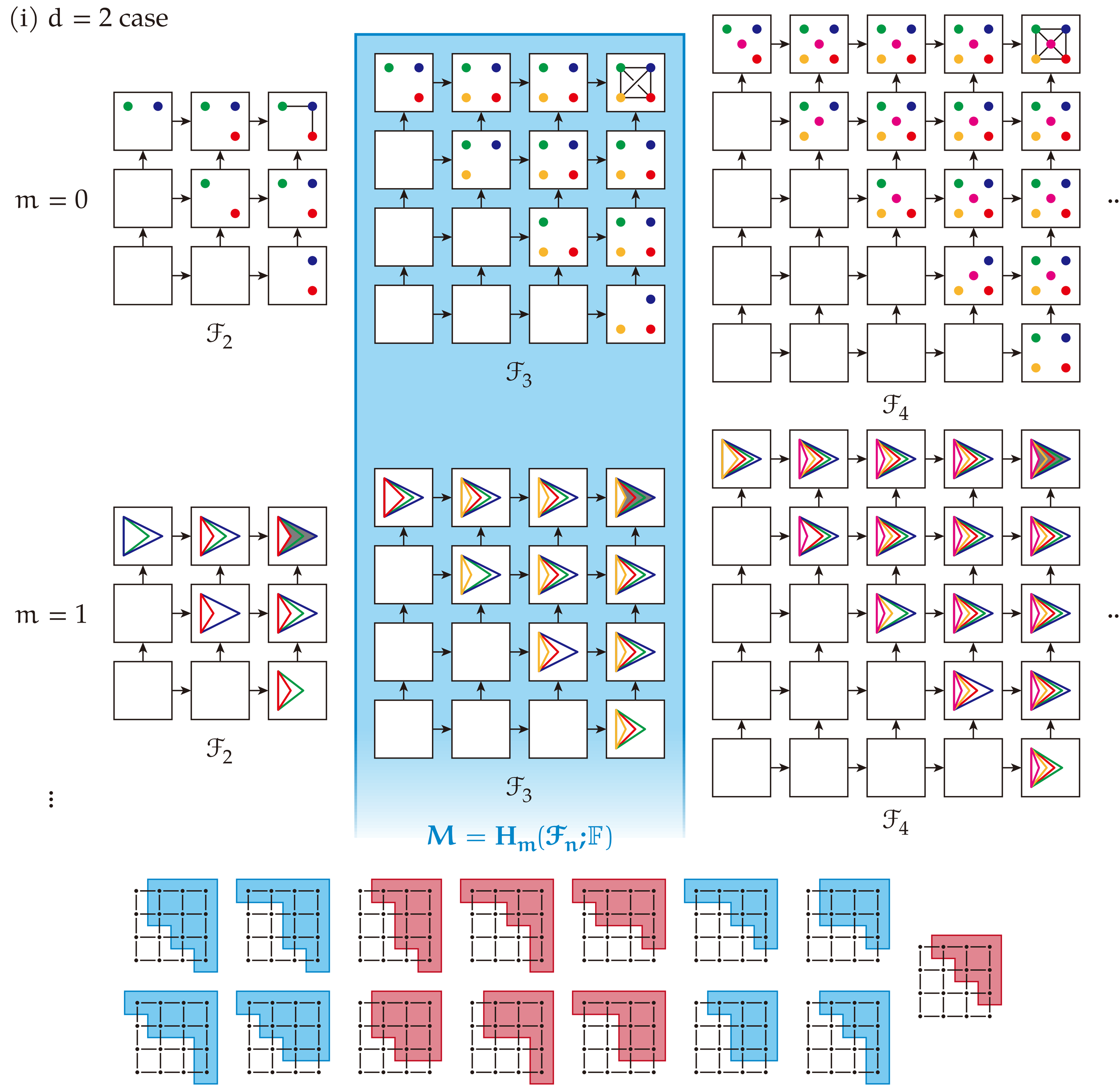
Notations

$\mathrm{Int}(P)$: the set of intervals (connected, convex subsets) of a poset P
dgm_m : the **m-th GPD**; the GPD obtained via $H_m(-;\mathbb{F})$
$\|\mathrm{dgm}_m(\mathcal{F})\|$: the **size** of the m-th GPD of a filtration $\mathcal{F}$
$\mathbb{I}_{[a,b]}$: the interval module on $[a,b]$

Main Results

Theorem 1. *For filtrations $\mathcal{F}$ over $\mathbb{R}^d$ ($d \geqslant 2$), there does* **not** *exist a polynomial in the number of simplices in $\mathcal{F}$, that bounds $\|\mathrm{dgm}_m(\mathcal{F})\|$ for arbitrary $\mathcal{F}$.*

(i) $d = 2$ case

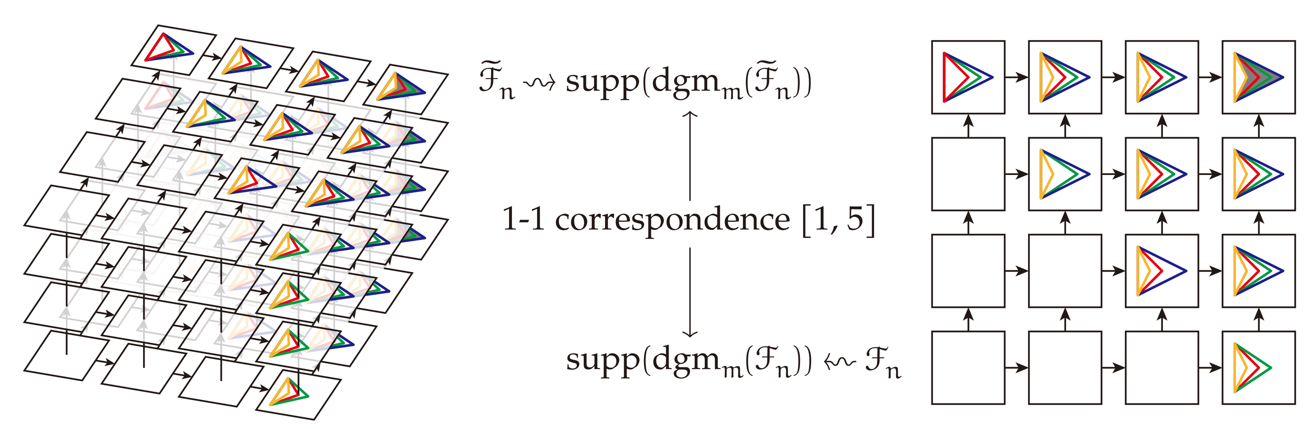


$\mathrm{dgm}_M(I) = +1$, $\mathrm{dgm}_M(I) = -1$

From the steps of the proof, we show that these intervals admit nonzero GPD values.
We have $\|\mathrm{dgm}_m(\mathcal{F}_n)\| \geqslant 2^{n+1} - 1$, which is super-polynomial in the number of simplices in $\mathcal{F}_n$.

Remark. $\|\mathrm{dgm}_m(\mathcal{F}_n)\| = 2^{n+2} - n - 3$ for $n \geqslant 3$.

(ii) $d > 2$ case

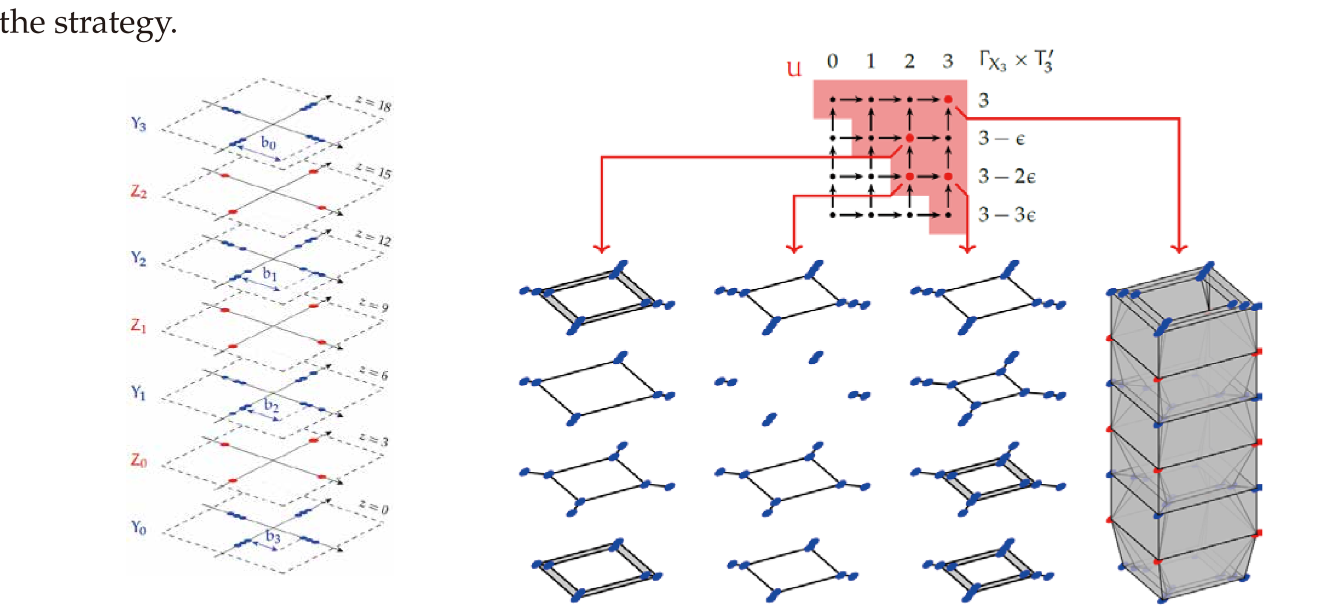


$$\begin{array}{l} \widetilde{M} := H_m(\widetilde{\mathcal{F}}_n;\mathbb{F}), \\ M := H_m(\mathcal{F}_n;\mathbb{F}) \end{array} \implies \mathrm{dgm}_{\widetilde{M}}(I') = \begin{cases} \mathrm{dgm}_M(I), & \text{if } I' = I \times [n]^{d-2} \text{ for } I \in \mathrm{Int}([n]^2) \\ 0, & \text{otherwise.} \end{cases}$$

We have $\|\mathrm{dgm}_m(\widetilde{\mathcal{F}}_n)\| = \|\mathrm{dgm}_m(\mathcal{F}_n)\|$, while the number of simplices in $\widetilde{\mathcal{F}}_n$ and $\mathcal{F}_n$ are the same.

Theorem 2. *For sublevel-Rips (sublevel-Čech/degree-Rips/degree-Čech) filtrations $\mathcal{F}$, there does* **not** *exist a polynomial in the number of 0-simplices in $\mathcal{F}$, that bounds $\|\mathrm{dgm}_1(\mathcal{F})\|$ for arbitrary $\mathcal{F}$.*

Considering sublevel-Rips filtrations, we construct a sequence $\{(X_n, \gamma_n)\}$ of metric spaces and augmented functions. The resulting sequence of sublevel-Rips filtrations resembles the sequence we have built in Theorem 1. We draw on the results of [7] to ensure the adaptation of the strategy.



A metric space X_3 — A part of the sublevel-Rips filtration of (X_3, γ_3)

For all the other types of filtrations, we apply a similar method.

Implication

Proposition. *For an arbitrary $[n]^d$-module M ($d \geqslant 2$), computing dgm_M via the Möbius inversion of the GRI of M requires at least $\Theta\left(2^{\binom{n+d-1}{d-1}}\right)$ time in general (even the GRI is given), even for computing $\mathrm{dgm}_M(I)$ for an interval I of $[n]^d$.*

The case of $d = 2$ is the direct consequence of the proof of Theorem 1. For $d > 2$, a slight modification of the proof of Theorem 1 implies the result.

$$\boxed{\begin{array}{c}\text{Persistence Module}\\ M : P \to \mathbf{vec}_{\mathbb{F}}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{GRI}\\ \mathrm{rk}_M : \mathrm{Int}(P) \to \mathbb{Z}_{\geqslant 0}\end{array}} \xrightarrow{\text{Möbius inversion}} \boxed{\begin{array}{c}\text{GPD}\\ \mathrm{dgm}_M : \mathrm{Int}(P) \to \mathbb{Z}\end{array}}$$

It suffices to perform the Möbius inversion on $\mathrm{supp}(\mathrm{dgm}_M)$ [5], but this computation is costly due to the size of $\mathrm{supp}(\mathrm{dgm}_M)$. Even for computing $\mathrm{dgm}_M(I)$ for a single interval I, there exists an interval which requires adding $\Theta\left(2^{\binom{n+d-1}{d-1}}\right)$ nonzero integers to compute it.

Conclusion

Summary

- The size of the GPD can be super-polynomial in the number of simplices in a given multi-parameter filtration, even if a given bifiltration is from well-known constructions.
- Even if the GRI is given, the computation of the GPD using the Möbius inversion of the GRI can be intractable.

Future directions

- Developing an output-sensitive algorithm [10]
- Efficiently approximating the GPD [3]

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