# Super-Polynomial Growth of the Generalized Persistence Diagram

Full version - arXiv:2412.04889

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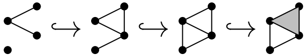
(1-parameter) Persistent Homology v. Multi-parameter Persistent Homology

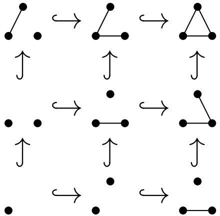
$\mathcal{F}$ :



$$\Downarrow$$

$$\mathrm{H}_0 : \quad \mathbb{F}^2 \longrightarrow \mathbb{F} \longrightarrow \mathbb{F} \longrightarrow \mathbb{F}$$

$$\mathrm{H}_1 : \quad 0 \longrightarrow \mathbb{F} \longrightarrow \mathbb{F}^2 \longrightarrow \mathbb{F}$$



$\Rightarrow$ Persistence module over a **poset**

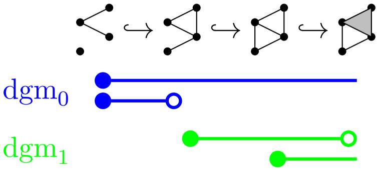
# Introduction

Persistent Homology v. Multi-parameter Persistent Homology

**Persistence Diagram**
(equivalently, **Barcode**)

[Edelsbrunner, Letscher, Zomorodian '02],

[Carlsson, Zomorodian '05]




$\mathrm{dgm}_m$ denotes the PD obtained via $\mathrm{H}_m(-;\mathbb{F})$.

?

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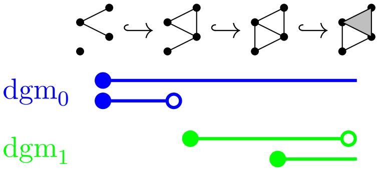


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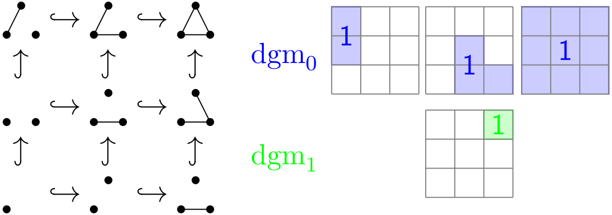
**Generalized Persistence Diagram (GPD)**

[Patel '16], [Kim, Mémoli '19],
[Asashiba, Escolar, Nakashima, Yoshiwaki '19]




Here, $\mathrm{dgm}_m$ denotes the GPD obtained via $\mathrm{H}_m(-;\mathbb{F})$.

# Introduction

Persistent Homology v. Multi-parameter Persistent Homology

**Persistence Diagram**

"viewed as the function"

$$\mathrm{dgm}_M([a,b]) = \text{mult. of } \mathbb{I}_{[a,b]} \text{ in } M$$

[Edelsbrunner, Letscher, Zomorodian '02], [Carlsson, Zomorodian '05]

**Rank Invariant**

$$\mathrm{rk}_M([a,b]) = \mathrm{rank}(M_a \to M_b)$$

[Carlsson, Zomorodian '09]

$$\Rightarrow\ \mathrm{rk}_M([a,b]) = \sum_{\substack{J \in \mathrm{Int}(\mathbb{Z}) \\ J \supseteq [a,b]}} \mathrm{dgm}_M(J)$$

**Generalized Rank Invariant (GRI)**

$$\mathrm{rk}_M : \mathrm{Int}(P) \to \mathbb{Z}_{\geq 0}$$

$$\mathrm{rk}_M(I) = \mathrm{rank}(\varprojlim M|_I \to \varinjlim M|_I)$$

[Kinser '08], [Kim, Mémoli '19], [AENY '19], [Chamber, Letscher '19]

The **GPD** is defined as the **Möbius inverse** of the GRI on intervals.

$$\mathrm{rk}_M(I) = \sum_{\substack{J \in \mathrm{Int}(P) \\ J \supseteq I}} \mathrm{dgm}_M(J)$$

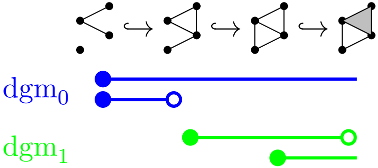
[Patel '16], [Kim, Mémoli '19], [AENY '19]

Intuitively, the GRI counts how many features persist throughout an interval, and the GPD tries to count how many features persist **exactly** on the interval (not beyond).

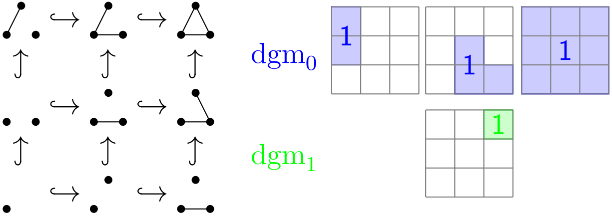
# Introduction

Persistent Homology v. Multi-parameter Persistent Homology

## Persistence Diagram




## Generalized Persistence Diagram (GPD)

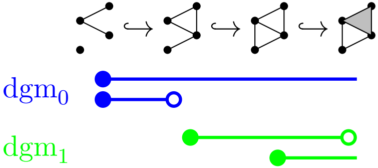



**Remark**

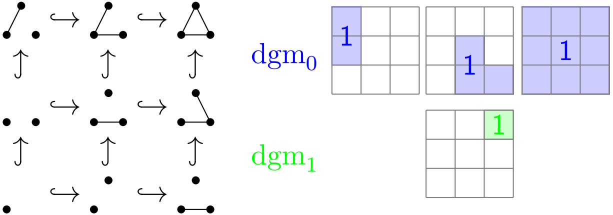
For a simplicial filtration $\mathcal{F}$ containing $N$ simplices, then, for each $m \in \mathbb{Z}_{\geq 0}$, $\mathrm{dgm}_m(\mathcal{F})$ contains **at most** $N$ bars.

Persistent Homology v. Multi-parameter Persistent Homology

**Persistence Diagram**




**Generalized Persistence Diagram (GPD)**

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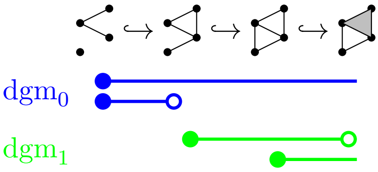
**Question**

What about the multi-parameter case?
Is there an analogy with the 1-parameter setting?

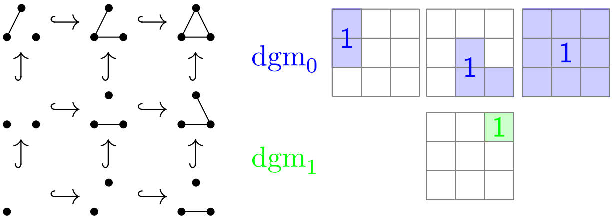
# Introduction

Persistent Homology v. Multi-parameter Persistent Homology

**Persistence Diagram**




**Generalized Persistence Diagram (GPD)**




### Remark

For a simplicial filtration $\mathcal{F}$ containing $N$ simplices, then, for each $m \in \mathbb{Z}_{\geq 0}$, $\mathrm{dgm}_m(\mathcal{F})$ contains **at most** $N$ bars.

### Question

What about the multi-parameter case?
Is there an analogy with the 1-parameter setting?

And our answer is **no**.

# Main Theorem 1

**Theorem (Kim, Kim, L., SoCG 2025)**

*For filtrations $\mathcal{F}$ over $\mathbb{R}^d$ $(d \geq 2)$, there does* **not** *exist a polynomial in the number of simplices in $\mathcal{F}$, that bounds $||\mathrm{dgm}_m(\mathcal{F})||$ for arbitrary $\mathcal{F}$.*

$||\mathrm{dgm}_m(\mathcal{F})||$ denotes the **size** (the cardinality of the support) of the $m^{\text{th}}$ GPD.

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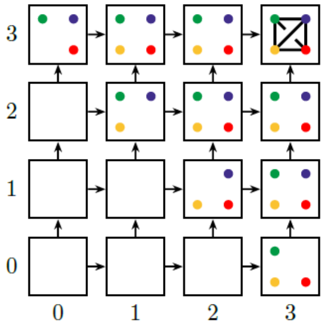
Proof. (i) $d = 2$ case




$m = 0,\ d = 2,\ \mathcal{F}_3\ (n = 3)$

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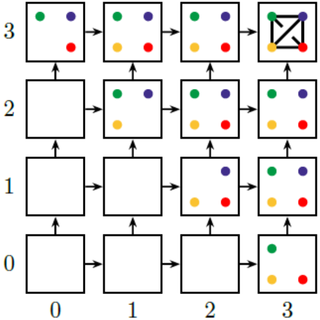


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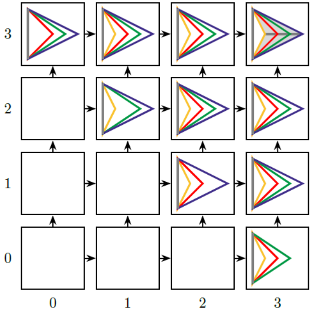
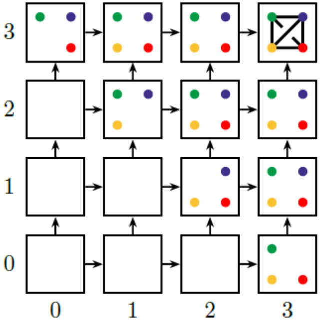
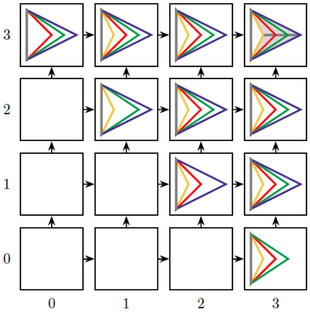
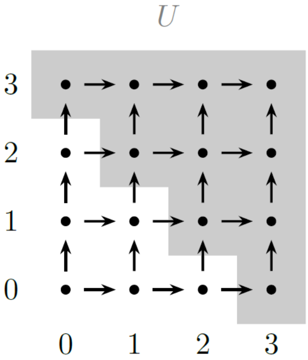


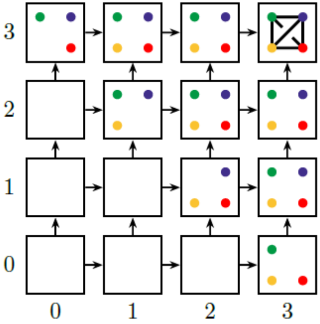

$m = 1,\ d = 2,\ \mathcal{F}_3\ (n = 3)$

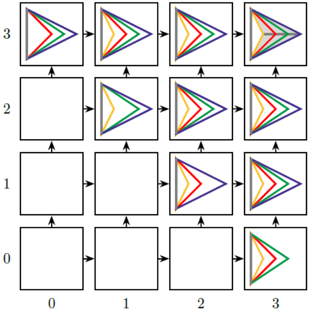


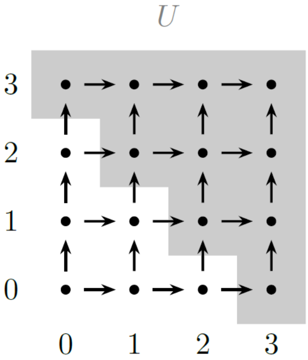




Step 1. If $J$ is not a proper subset of $U$, then $\mathrm{rk}_M(J) = 0$.



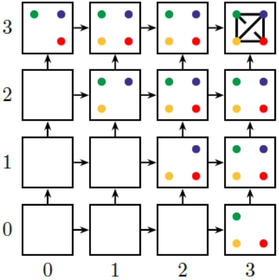



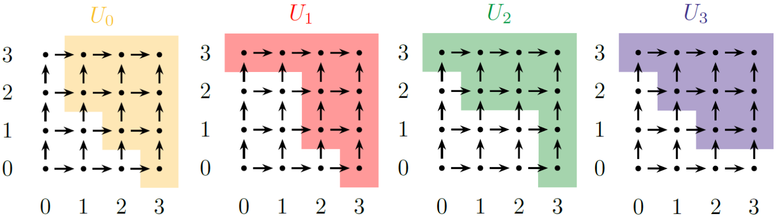

Step 1. If $J$ is not a proper subset of $U$, then $\mathrm{rk}_M(J) = 0$.

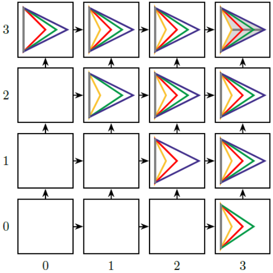
- None of the features persist over the entire $U$

$\Rightarrow \mathrm{dgm}_M(J) = 0$; don't need to consider the interval $J$

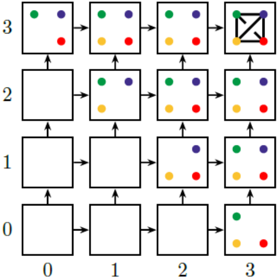
# Main Theorem 1

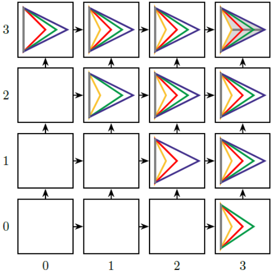


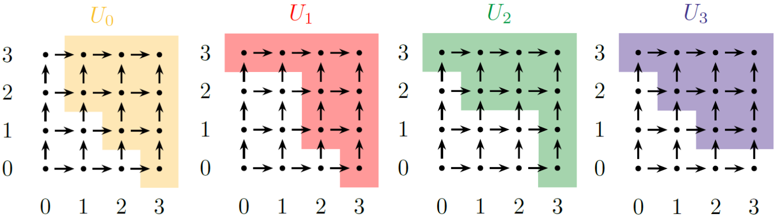




Step 2. For $i \in [n]$, $\mathrm{rk}_M(U_i) = 1 \;\Rightarrow\; \mathrm{dgm}_M(U_i) = 1$





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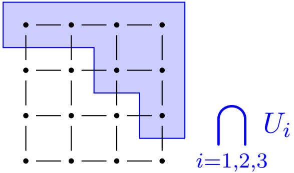
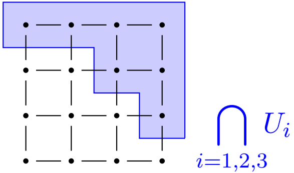
- Only the yellow feature persists throughout $U_0$.
- Similarly, for all other intervals $U_i$
- $1 = \mathrm{rk}_M(U_i) = \sum_{J \supseteq U_i} \mathrm{dgm}_M(J) = \mathrm{dgm}_M(U_i)$

  from Step 1

# Main Theorem 1




Step 3. For $\emptyset \neq S \subseteq [n]$, $\mathrm{dgm}_M\left(\bigcap_{i \in S} U_i\right) = (-1)^{|S|+1} \neq 0$.

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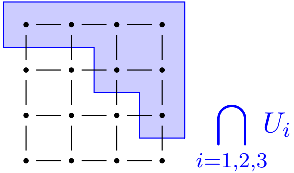
- The result comes from Step 1, 2, and a lemma [AENY '19].

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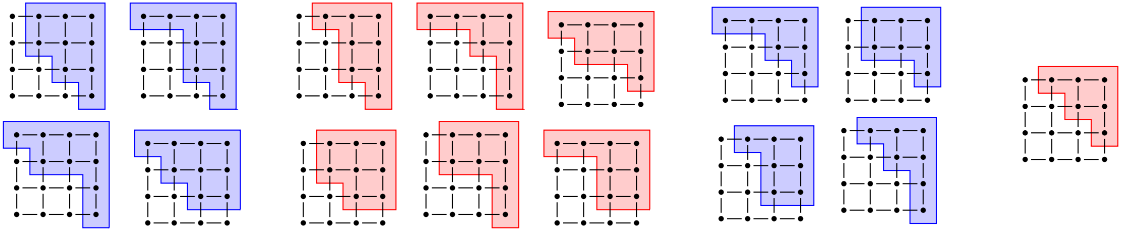
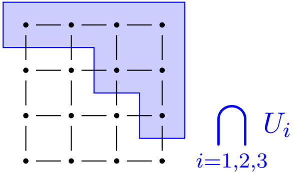


These are the intervals with nonzero GPD values identified through the previous steps.

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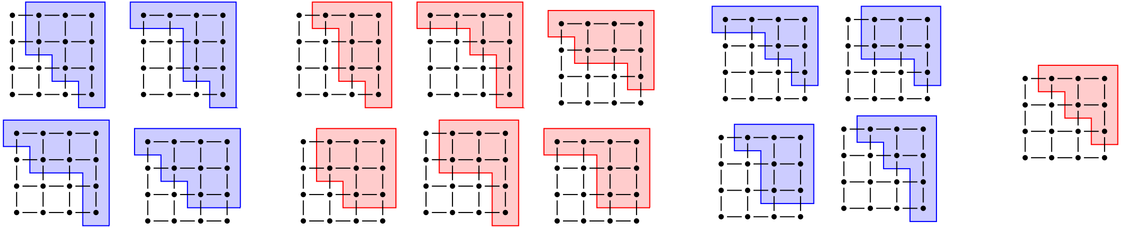

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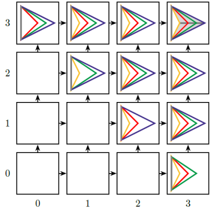
**We have** $\boldsymbol{||\mathrm{dgm}_m(\mathcal{F}_n)|| \geq 2^{n+1} - 1 \notin O(n^{(m+2)k}) = O(N^k)}$.

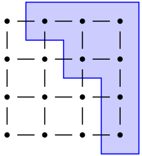
# Main Theorem 1

(ii) What if $d > 2$?

$\mathcal{F}_n$



$$\rightsquigarrow \quad \mathrm{supp}(\mathrm{dgm}_m(\mathcal{F}_n)) \quad \ni \quad I$$



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(ii) What if $d > 2$?

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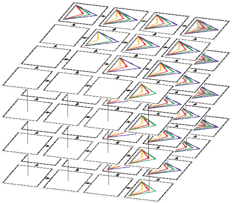
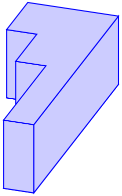
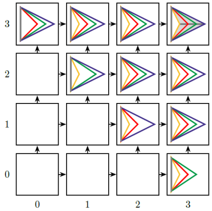


$\pi$

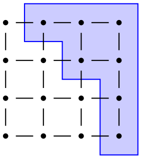
$\pi^{-1}(I)$



$\mathcal{F}_n$




$\rightsquigarrow \quad \mathrm{supp}(\mathrm{dgm}_m(\mathcal{F}_n)) \quad \ni \quad I$



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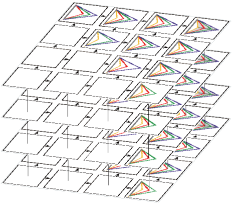
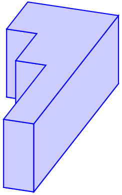


$$\rightsquigarrow \quad \operatorname{supp}(\operatorname{dgm}_m(\mathcal{F}'_n)) \quad \ni \quad \pi^{-1}(I)$$



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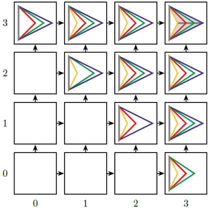
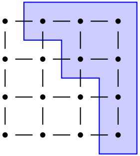
[Botnan, Oppermann, Oudot, '24]
[Clause, Kim, Mémoli, '24] $\Longrightarrow$ correspondence

$\mathcal{F}_n$

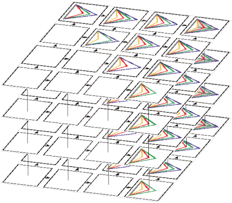
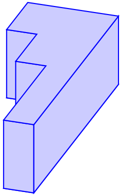


$$\rightsquigarrow \quad \operatorname{supp}(\operatorname{dgm}_m(\mathcal{F}_n)) \quad \ni \quad I$$



# Main Theorem 1

(ii) What if $d > 2$?

$\mathcal{F}'_n$  $\rightsquigarrow$ $\mathrm{supp}(\mathrm{dgm}_m(\mathcal{F}'_n)) \ni \pi^{-1}(I)$ 

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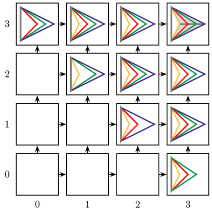
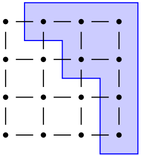
$\mathcal{F}_n$ 

$\rightsquigarrow$ $\mathrm{supp}(\mathrm{dgm}_m(\mathcal{F}_n)) \ni I$ 

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# Main Theorem 2

**Question**

Can GPDs with super-polynomial size also arise from common 2-parameter filtrations?

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Can GPDs with super-polynomial size also arise from common 2-parameter filtrations?

**Answer**

**Yes,** considering the $1^{\text{st}}$ GPD of a sublevel-Rips, a sublevel-Čech, a degree-Rips, or a degree-Čech filtration.

These 4 types of filtration are well-known constructions of 2-parameter filtrations.
We will take a look at the case of sublevel-Rips filtrations.

# Main Theorem 2: Sublevel-Rips case

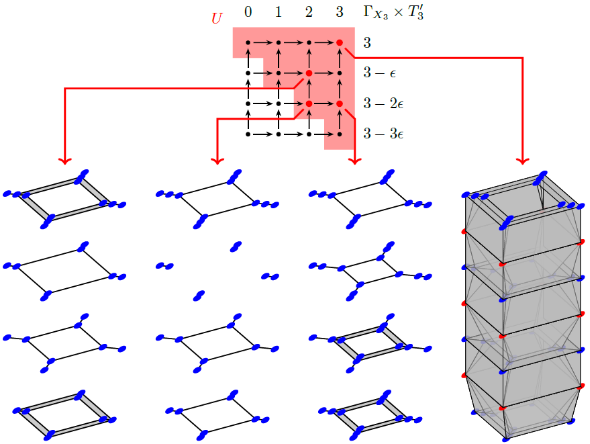
**Theorem (Kim, Kim, L., SoCG 2025)**

*For sublevel-Rips filtrations $\mathcal{F}$, there does* **not** *exist a polynomial in the number of simplices in $\mathcal{F}$, that bounds $\|\mathrm{dgm}_1(\mathcal{F})\|$ for any $\mathcal{F}$.*

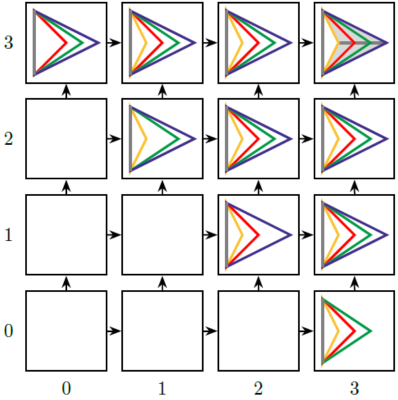
**Definition [Carlsson, Zomorodian '09]**

Given a metric space $(X, \mathrm{d})$ and a function $\gamma : X \to \mathbb{R}$, the **sublevel-Rips** filtration is a filtration $\mathcal{F}$ on $\mathbb{R}^2$, which for each $(r, s) \in \mathbb{R}^2$, $\mathcal{F}_{(r,s)}$ is the Vietoris-Rips complex formed by a sublevel set $\{x \in X : \gamma(x) \leq r\}$, at scale $s$.

The (restricted) filtration when $n = 3$.




cf. $\mathcal{F}_3$ in Main theorem 1




We adopt a similar strategy with Main theorem 1, by drawing on the results of [Gülen, McCleary '23].

# Implication

**Corollary (Kim, Kim, L., SoCG 2025)**

*For an arbitrary* $[n]^d$*-module* $M$ $(d \geq 2)$*, computing* $\mathrm{dgm}_M(I)$ *via the Möbius inversion of the GRI of* $M$ *requires at least* $\Theta(2^n)$ *in time in general (even the GRI is given).*

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$$\boxed{\begin{array}{c}\text{Persistence module}\\ M : P \to \mathbf{vec}_{\mathbb{F}}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{GRI}\\ \mathrm{rk}_M : \mathrm{Int}(P) \to \mathbb{Z}_{\geq 0}\end{array}} \xrightarrow{\textbf{MI}} \boxed{\begin{array}{c}\text{GPD}\\ \mathrm{dgm}_M : \mathrm{Int}(P) \to \mathbb{Z}\end{array}}$$

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[Clause, Kim, Mémoli, '24]

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|---|---|---|---|---|
| | $\Uparrow$ | | $\Uparrow$ | |
| | This computation is costly | | It suffices to perform on $\mathrm{supp}(\mathrm{dgm}_M)$<br>[Clause, Kim, Mémoli, '24]<br>$\Uparrow$<br>This computation is **also** costly | |

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**Summary**

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Thank you for your attention!